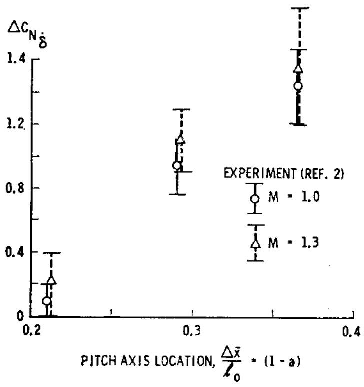

Fig. 6 Effect of pitch axis location on damping force for a 12.5° sharp cone.

tion effects. The problem is to determine how much of this increase in damping would have been observed in the absence of a sting. If one assumes that the transition-induced force is located at the base, the increased $\Delta C_{N\dot{\delta}}$ corresponding to the nonlinear increase in $C_{m\dot{\delta}}$ at $\alpha = 0$ in Fig. 2 is as shown in Fig. 6. The increase of $\Delta C_{N\dot{\delta}}$ with increasing $\Delta\bar{x}$ and consequently increased sting plunging, must be caused by sting interference. Correcting Wehrend's data for this sting interference gives vastly improved agreement with the various theories (Fig. 1). The other set of experimental data[9] is not affected by transition or support interference. Yanagizawa's results[9] were obtained using a magnetically suspended half model. It is, of course, possible that the wall boundary layer caused some interference, but another explanation for the difference between the two sets of "support-interference-free" data is the fact that Wehrend's data should be high due to the damping effect of boundary-layer transition per se,[7] in the absence of a sting.‡

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‡ In view of the present note, the results in Ref. 7 will have to be re-examined for the possibility of dynamic support interference.

First-Excursion Failure of Randomly Excited Structures: II

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In a recent paper[1] the probability for a randomly excited structure to survive a given service time interval $(0,t]$ without suffering a first-excursion failure was considered under rather general conditions. It was shown that the probability of survival involved the statistics of the excursion rate $N(t)$ and that an approximation to this probability derived from a necessary condition for "nonapproaching" excursions and requiring only the knowledge of the ensemble mean $E[N(t)]$ and the correlation function $E[N(t_1)N(t_2)]$ was better than the Poisson estimate (assuming independent excursions) when compared with available simulation results.[2]

The present Note is devoted to another scheme of approximation for the survival probability based on a maximum entropy principle proposed by Jaynes.[3,4] It is known that entropy is a measure of uncertainty in a random variable; therefore, an estimate of a probability distribution corresponding to the maximum entropy is the least biased estimate. This new scheme of approximation appears also to produce still better results than the nonapproaching excursion estimates.

Least Biased Estimates of Probability

It can be shown[3,4] that if our knowledge about a discrete random variable Z is limited to a number of statistics

$$E[\phi_j(Z)] = \psi_j, \quad j = 1,2,\ldots,M \tag{1}$$

then the probability $P_k = \text{Prob}\,[Z = z_k]$ corresponding to the maximum entropy is given by

$$P_k = \exp\left[-\lambda_0 - \sum_{j=1}^{M} \lambda_j \phi_j(z_k)\right] \tag{2}$$

where the constants λ are related as follows:

$$\lambda_0 = \ln \sum_k \exp\left[-\sum_j \lambda_j \phi_j(z_k)\right] \tag{3}$$

$$\frac{\partial \lambda_0}{\partial \lambda_j} = -\psi_j, \quad j = 1,2,\ldots,M \tag{4}$$

Let $\eta(t)$ be the total number of times that a structural response $X(\tau)$, $0 < \tau \le t$, passes outside the safety region $-a \le x \le b$, and let the entropy of $\eta(t)$ be maximized. Since $\eta(t) = \int_0^t N(\tau)d\tau$ an application of ensemble average gives

$$E[\eta(t)] = \int_0^t g_1(\tau)d\tau \tag{5}$$

where $g_1(\tau) = E[N(\tau)]$. From Eq. (3),

$$\lambda_0 = \ln \sum_{k=0}^{\infty} \exp(-\lambda_1 k) = -\ln(1 - e^{-\lambda_1}) \tag{6}$$

Differentiate the above with respect to λ_1

$$\frac{d\lambda_0}{d\lambda_1} = e^{-\lambda_1}(e^{-\lambda_1} - 1)^{-1} = -\int_0^t g_1(\tau)d\tau, \tag{7}$$

where, in the last step, use has been made of Eq. (4) with $\int_0^t$

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$g_1(\tau)d\tau$ substituting for ψ_1. Eliminating λ_1 from Eqs. (6) and (7), we obtain

$$\lambda_0 = \ln\left[1 + \int_0^t g_1(\tau)d\tau\right] \tag{8}$$

The approximate probability of survival corresponding to the most uncertain $\eta(t)$ and subject to the constraint of Eq. (5) is thus

$$P_0(t) = e^{-\lambda_0} = \left[1 + \int_0^t g_1(\tau)d\tau\right]^{-1} \tag{9}$$

It is interesting to note that this approximate probability is bounded below by the Poisson estimate, Eq. (11) of Ref. 1.

Suppose that in addition to the constraint Eq. (5), we also have the information

$$E[\eta^2(t)] = \int_0^t\int_0^t f_2(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{10}$$

where $f_2(\tau_1,\tau_2) = E[N(\tau_1)N(\tau_2)]$. Then

$$\lambda_0 = \ln\sum_{k=0}^{\infty} \exp(-\lambda_1 k - \lambda_2 k^2) \tag{11}$$

The summation on the right-hand side of Eq. (11) may be approximated by the following integral

$$\int_0^\infty \exp(-\lambda_1 k - \lambda_2 k^2)dk = \frac{1}{2}(\pi/\lambda_2)^{1/2}(1 - \text{erf}x)\exp x^2 \tag{12}$$

where $x = \lambda_1/(2\lambda_2^{1/2})$ and erf$x$ is the error function. Application of Eq. (4) and simplification lead to

$$\lambda_2 E[\eta^2(t)] = \tfrac{1}{2} + x^2 + x\gamma \tag{13}$$

and

$$E[\eta^2(t)]/\{E[\eta(t)]\}^2 = \frac{1}{2}(\gamma - x)^{-2} - x(\gamma - x)^{-1} \tag{14}$$

where

$$\gamma = (\pi)^{-1/2}(1 - \text{erf}x)^{-1}\exp(-x^2) \tag{15}$$

The procedure of numerical calculation is as follows: 1) determine x from the ratio $E[\eta^2(t)]/\{E[\eta(t)]\}^2$ using Eq. (14); 2) compute λ_2 from Eq. (13); and 3) compute the probability of survival estimate from

$$P_0(t) = \exp(-\lambda_0) = [\tfrac{1}{2}(\pi/\lambda_2)^{1/2}(1 - \text{erf}x)\exp x^2]^{-1} \tag{16}$$

It is of interest to note that the right hand side of Eq. (14) is monotonically increasing with x and is bounded between 1 and

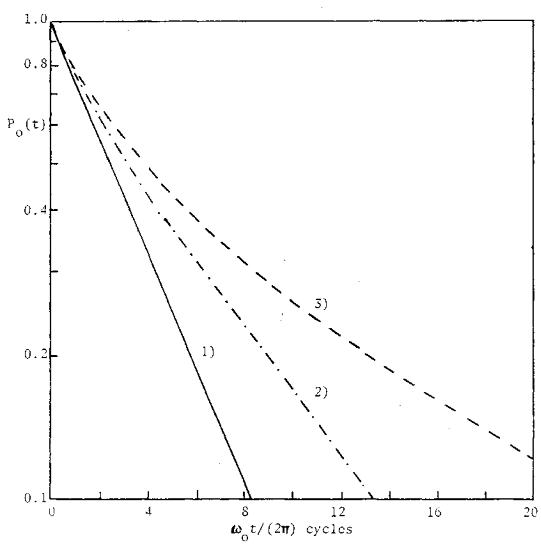

Fig. 1 Probability of survival for simple oscillator with $\zeta = 0.03$ and failure bounds $\pm 2\sigma$; 1) Poisson excursions, 2) nonapproaching excursions, 3) first-order maximum entropy estimate.

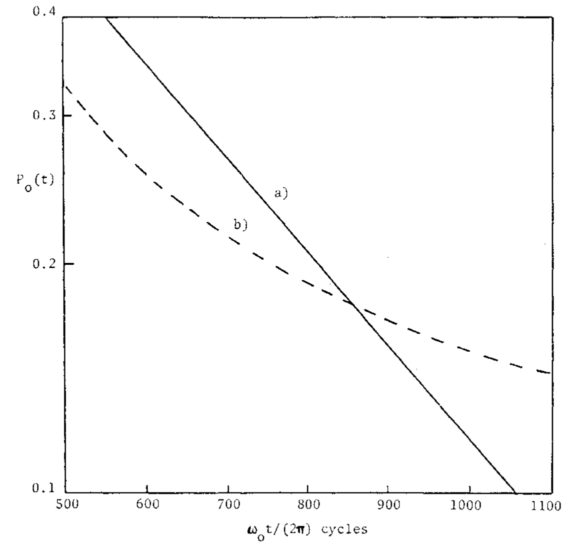

Fig. 2 Probability of survival for simple oscillator with $\zeta = 0.02$ and failure bounds $\pm 3.5\sigma$; a) second-order maximum entropy estimate, b) first-order maximum entropy estimate.

2. The lower bound does not impose a restriction since it agrees with Schwarz's inequality. However, the upper bound is an additional limitation; it implies that the above procedure does not apply in the case $E[\eta^2(t)]/\{E[\eta(t)]\}^2 > 2$.

Stationary Structural Response

Now if the structural response $X(\tau)$ is stationary, so also is the excursion rate process $N(\tau)$. For such a case $g_1(\tau) = E[N(\tau)]$ is just a constant and $f_2(\tau_1,\tau_2) = E[N(\tau_1)N(\tau_2)]$ is just a function of $\tau = \tau_1 - \tau_2$. Then Eq. (9) reduces to

$$P_0(t) = (1 + g_1 t)^{-1} \tag{9a}$$

The computation of the second-order estimate, Eq. (16), requires a knowledge of $E[\eta^2(t)]$. In the stationary case we only need carry out the computation of $E[\eta^2(t)]$ to a moderate t in order that its value at large t can be projected. Now the integrand in Eq. (10) is an even function of $\tau = \tau_1 - \tau_2$. Thus

$$E[\eta^2(t)] = \int_0^t\int_0^t f_2(\tau_1 - \tau_2)d\tau_1 d\tau_2 = 2t\int_0^t f_2(\tau)\left(1 - \frac{\tau}{t}\right)d\tau \tag{17}$$

Substituting $f_2(\tau) = g_2(\tau) + g_1^2$ we obtain

$$E[\eta^2(t)] = 2t\int_0^t g_2(\tau)\left(1 - \frac{\tau}{t}\right)d\tau + g_1^2 t^2 \tag{18}$$

The advantage of dealing with $g_2(\tau)$ instead of $f_2(\tau)$ is as follows. We note that as τ increases $g_2(\tau)$ vanishes, whereas $f_2(\tau)$ tends to g_1^2. Furthermore, it is reasonable to assume that $g_2(\tau)$ is absolutely integrable on $(0,\infty)$. Thus

$$\lim_{t\to\infty}\int_0^t g_2(\tau)\left(1 - \frac{\tau}{t}\right)d\tau = \lim_{t\to\infty}\int_0^t g_2(\tau)d\tau = \frac{K}{2} \tag{19}$$

where K is a constant. Therefore, for sufficiently large t

$$E[\eta^2(t)] \approx Kt + g_1^2 t^2 \tag{20}$$

Equation (20) suggests a practical way to evaluate K by graphing $E[\eta^2(t)] - g_1^2 t^2$ vs t. As t increases, the plot becomes asymptotic to a straight line whose slope is, of course, K. The computer evaluation of $E[\eta^2(t)]$ may be discontinued as soon as this slope is clearly established.

Numerical Results and Discussion

Figure 1 shows the results obtained from the first-order formula, Eq. (9), for a single-degree-of-freedom linear oscillator under Gaussian white noise excitation and for rather low safety threshold levels $\pm 2\sigma$, where σ is the standard deviation of the stationary response. The Poisson and the nonapproaching excursion estimates are also plotted for comparison. Within the time range investigated, Eq. (9) gives a higher estimate than the corresponding Poisson and nonapproaching excursion estimates.

Figure 2 shows the results obtained from the second-order formula, Eq. (16), in the time region where the asymptotic form Eq. (20) for $E[\eta^2(t)]$ is applicable and for relatively high safety bounds $\pm 3.50\sigma$. Corresponding results computed from Eq. (9) are also plotted for comparison. The comparison reveals that using the most uncertain $\eta(t)$ as a criterion, a first-order estimate may not always be more conservative than a second-order estimate.

Another point of interest is that all the second-order estimates tend to straight lines when plotted on a semilogarithmic scale. This implies that these estimates tend to the exponential form, a general form conjectured by Mark[5] and verified by Crandall, et al.[2] by simulation. For example, for damping ratio $\zeta = 0.02$ and failure bounds $\pm 3.5\sigma$ the second order estimate of $P_0(t)$ tends to exp(-0.00279 natural cycles), suggesting an average first excursion time in the neighborhood of $1/0.00279 = 357$ natural cycles. The simulation results[2] for the same system have an average of 744 cycles.

Equations (9) and (16) were obtained from maximization of the entropy of $\eta(t)$. A different estimate for P_0 would result if maximization was associated with a random variable other than $\eta(t)$. Tribus[4] has shown that the maximum entropy estimate for P_0 with the knowledge of the average structural life, L, is given by

$$P_0(t) = \exp(-t/L) \tag{21}$$

However, no general method is available to compute L for a practical structural response. Note that Eq. (21) would agree with the Poisson estimate if L could be replaced by $1/g_1$. Clearly, $1/g_1$ is the average recurrence time which is a conditional average whereas L is an unconditional average. From a physical argument we can assert that $L \leq 1/g_1$. If, however, we are ignorant of any statistical relationship between different excursions, then the assumption of independent excursions should be the most unbiased one, and Eq. (21) would coincide with the Poisson estimate.

It is appropriate to comment on the general philosophy of a statistical analysis from the standpoint of maximum uncertainty. Since nearly all experimental measurements of a random phenomenon are directed toward the first- and the second-order moments, our knowledge of the physical situation is seldom beyond these levels. However, when theoretical computation requires an expression for the probability distribution, an analyst is forced to extrapolate from his limited knowledge. Such extrapolations are sometimes dictated by mathematical conveniences. Thus, the Gaussian distribution is often a favorite assumption because of its desirable mathematical properties. For example, a Gaussian structural response has been assumed in the computation of $g_1(t)$ and $f_2(t_1,t_2)$ reported herein. This procedure is quite justified, however, since the Gaussian random process is associated with the maximum entropy if our knowledge of the response is limited to the mean and the correlation functions. To estimate the structural reliability from the functions $g_1(t)$ and $f_2(t_1,t_2)$, we used, once again, the maximum entropy principle and arrived at Eqs. (9) and (16). Thus, Eqs. (9) and (16) were the results of repeated applications of the maximum entropy principle. These are the least prejudiced estimates subject to the respective constraints of statistical knowledge. The maximum entropy results are also expected to be conservative as supported by one example where the estimated average structural life is about one-half the simulation average.

Finally, it should be noted that although a single-degree-of-freedom system was chosen in the example, the present method applies to any structure as long as the functions $g_1(t)$ and $f_2(t_1,t_2)$ can be calculated.

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Flow with $M_\infty = 1$ Past Thin Airfoils

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It has been recognized for some time that the pressure distribution on thin nonlifting airfoils in steady two-dimensional flow with freestream Mach number M_∞ equal to, or near, unity may be calculated by means of the local linearization theory.[1–3] The purpose of this note is threefold: 1) to present some alternative, but equivalent, expressions for the pressure that are more convenient for numerical computation than those given heretofore, 2) to present results for the airfoils tested by Michel et al.[4,5] using the exact equations to describe the airfoil ordinates rather than close approximations which permit analytical solutions as done originally, and 3) to examine the discrepancies that appear near the trailing edge in nearly all comparisons of theoretical and experimental pressure distributions for $M_\infty = 1$ to determine whether they may be inherent in the inviscid flow, as opposed to the usual explanation that they are due to shock-wave boundary-layer interaction.

According to the local linearization theory, the pressure coefficient C_p, or its transonic similarity counterpart $\tilde{C}_p = [M_\infty{}^2(\gamma + 1)/\tau^2]^{1/3} C_p$, at an arbitrary point x on the surface of a thin nonlifting airfoil having ordinates Z and thickness-chord ratio $\tau = t/c$ in a flow with $M_\infty = 1$ and ratio of specific heats γ (for air, $\gamma = \frac{7}{5}$) is given by

$$\tilde{C}_p = -2\left\{\frac{3}{\pi}\int_{x^*}^{x}\left[\frac{d}{dx_1}\int_0^{x_1}\frac{d(Z/\tau)/d\xi}{(x_1-\xi)^{1/2}}\,d\xi\right]^2 dx_1\right\}^{1/3} = -2\left\{\frac{3}{\pi}\int_{x^*}^{x}[F(x_1)]^2\,dx_1\right\}^{1/3} \tag{1}$$

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